NEW YORK UNIVERSITY

Institute of Mathematical Sciences

Division of Electromagnetic Research

RESEARCH REPORT No. EM-86

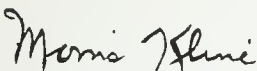# Forward Scattering of High-Frequency Plane Waves by a Sphere

GEORGE KEAR

NEW YORK UNIVERSITY

Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. EM-86

FORWARD SCATTERING OF HIGH-FREQUENCY PLANE WAVES BY A SPHERE

George Kear

George Kear

George Kear

Morris Kline

Morris Kline
Project Director

New York, 1955

Abstract

An expression for the scattered wave as an expansion in terms of radial eigenfunctions is obtained. The total wave may be expressed in terms of radial eigenfunctions directly but the incident wave cannot. Instead, the incident wave is first expressed by a contour integral. Then a change of variable is introduced and the resulting integral is approximated by the Euler-Maclaurin sum rule. This results in a series for the incident wave in terms of radial functions plus an integral and correction terms. When this is subtracted from the total wave a finite series in terms of radial functions is obtained, and the integral and correction terms are easily evaluated.

Table of Contents

1. Introduction

This paper is concerned with obtaining the amplitude of plane waves scattered in the forward direction by a sphere whose radius is large compared to the wavelength. Both Dirichlet (vanishing function) and Neumann (vanishing normal derivative) boundary conditions will be treated.

The procedure used for this problem is applicable to many electro-magnetic and quantum mechanical problems dealing with waves scattered by spherically stratified regions with various boundary conditions.

The description for the scattered wave in terms of partial waves has practical limitations for short wavelengths because of poor convergence. For the total wave this difficulty has been overcome by means of an alternate re-presentation characterized by an expansion in terms of radial eigenfunctions (see Friedman, Franz[1]) rather than angular eigenfunctions which appear in the partial wave solution. The purpose of this paper is to show a method for expressing also the scattered wave through the radial representation.

These two alternate forms for the total wave exhibit distinctly dif-ferent convergence properties. The expansion in terms of angular eigenfunctions is more rapidly convergent for plane-wave scattering and large wavelengths. On the other hand, the radial representation is more rapidly convergent for small wavelengths when the source is close to the sphere (cf. Marcuvitz[2], p. 310). Since the scattered wave alone is produced by sources (induced by the incident wave) directly on the sphere, the radial expansion promises to be more rapidly convergent for the scattered wave for short wavelengths.

The incident wave cannot be subtracted directly from the total wave in its radial form, because the radial eigenfunctions are constructed to vanish

on the surface of the sphere, in keeping with the boundary conditions. Therefore, no combination of these eigenfunctions can duplicate the value of the plane wave on the surface.

Instead, the incident field is represented first by its expansion in terms of partial waves; this expansion can then be expressed as a contour integral, by applying a Watson transformation. Through the use of the Euler-Maclaurin sum rule, a portion of the integral may be represented by a series of radial eigen-functions. When the result is compared with the sum for total wave, this radial representation does indeed show the expected convergence properties. Only a few terms in the series for the total wave needed be retained, and the remaining portion of the integral for the plane wave is easily evaluated.

The final numerical values for the first order frequency dependent cor-rection to the total cross section for both boundary conditions agree essential-ly with those obtained by other authors (cf. Wu and Rubinow, White[7]). The method used here, however, is not primarily dependent on computational methods and would appear to have more physical significance.

2. Formulation of the problem

Consider a sphere of radius a whose center lies at the origin of the set of spherical coordinates as shown in Fig. 1
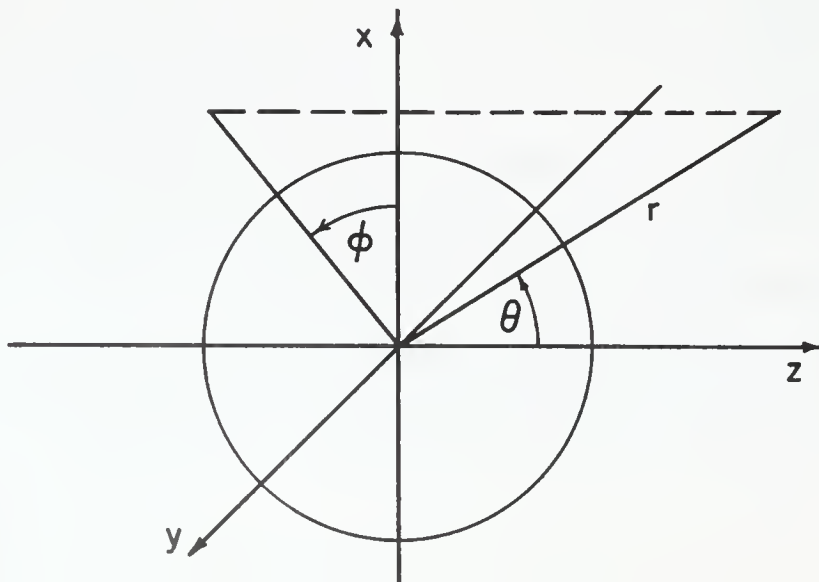

Figure 1

Incident on the sphere and propagating in the positive z-direction is a plane wave u_i of wave number k:

$$(2.1) \qquad u_i = e^{ikr \cos \theta} .$$

The reduced wave equation that describes the total field for the corresponding acoustic, electromagnetic, or quantum mechanical problem is given by

$$(2.2) \qquad \nabla^2 u + k^2 u = 0 ;$$

for this particular problem we specify the impedance boundary condition

$$
\begin{aligned}
\frac{\partial u}{\partial r} - kzu &= 0 \qquad \text{for } r = a , \\
(2.3) \\
\frac{\partial u}{\partial r} &\to iku \qquad \text{as } r \to \infty .
\end{aligned}
$$

The solution for this problem expressed as an expansion in terms of Legendre functions has been previously obtained (Friedman, Franz[1]):

$$(2.4) \quad u = \sum_{n=0}^{\infty} j_n(kr)i^n(2n+1)P_n(\cos \theta) - \sum_n \frac{A_n(ka)}{B_n(ka)} h_n^{(1)}(kr)i^n(n+1/2)P_n(\cos \theta) ,$$

where $A_n = 2j_n'(ka) - z\, 2j_n(ka)$ and $B_n = h_n^{(1)'}(ka) - z\, h_n^{(1)}(ka)$. Note that for $r = a$, equation (2.4) satisfies the condition prescribed by (2.3).

Note also that the first sum in (2.4) is the expansion for the plane wave in terms of the P_n, namely (cf. Stratton[3])

$$(2.5) \qquad u_i = e^{ikr \cos \theta} = \sum j_n(kr)i^n(2n+1)P_n(\cos \theta) .$$

Therefore the remaining sum in equation (2.4) represents the scattered wave.

However, equation (2.4) is not the only means of expressing the total field. An alternative expression for u may also be written:

$$(2.6) \quad u = i \sum_n \left[\frac{A_{\nu_n}(ka)}{\frac{\partial}{\partial \nu} B_\nu(ka)} \right]_{\nu=\nu_n} \frac{(\nu_n + \frac{1}{2})\pi \exp\left[-\frac{i\pi}{2}(\nu_n + 1)\right]}{\sin \nu_n \pi} P_{\nu_n}(\cos \theta) h_{\nu_n}^{(1)}(kr) \; .$$

In this equation, the values ν_n are obtained from those values of ν that satisfy

$$(2.7) \quad B_{\nu_n} \equiv h_{\nu_n}^{(1)'}(ka) - z\, h_{\nu_n}^{(1)}(ka) = 0 \; .$$

Equation (2.6) may be arrived at by applying the Watson transformation (see Friedman, Franz[1]) to (2.4) or by a technique using separation of variables and utilizing the properties of the Green's functions and eigenfunctions of r and (θ)(see Marcuvitz[2], pp. 309 ff.).

We note that $h_{\nu_n}^{(1)}(kr)$ satisfy the radial equation (2.2), that they satisfy the boundary condition at r = a as indicated by (2.7), and that they are outgoing as r becomes large. Therefore $h_{\nu_n}^{(1)}(kr)$ are the radial eigenfunctions, and (2.6) is the above-mentioned alternate expansion of the solution in terms of the radial eigenfunctions. The convergence properties of these two representations (2.6) and (2.4) have already been mentioned in the Introduction.

3. Subtraction of the plane wave

We wish to obtain a representation for the scattered wave that will exhibit the convergence properties of an expansion in terms of radial modes.

Since the plane wave cannot be split off directly from (2.6), as it was from (2.4), it must be subtracted in some other manner. To this end, we first represent the plane wave given by (2.5) by a contour integral. Accordingly, we may write u_i in the form

$$(3.1) \qquad u_i = \frac{1}{2\pi i} \oint \frac{j_\nu(kr)(2\nu+1)\, e^{i\nu\pi/2}\, P_\nu(\cos\theta)}{\frac{1}{\pi}\, e^{i\nu\pi}\, \sin\nu\pi}\, d\nu \quad,$$

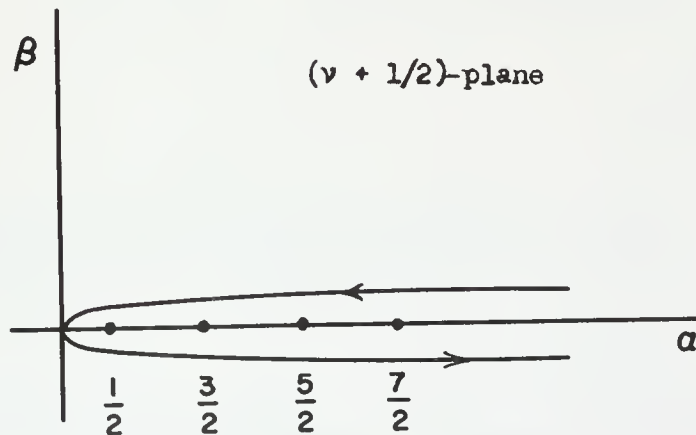where the residues reduce to the series (2.5) when the contour is taken as shown in Fig. 2.

Figure 2

The lower branch of the contour may be reflected by introducing

$$(3.2) \qquad \nu = -\nu' - 1 \quad,$$

because then

$$(3.3) \quad \int_{\nu+1/2=0}^{\infty} 2j_\nu \frac{(\nu+1/2)\pi}{\sin \nu\pi} e^{-i\nu\pi/2} P_\nu d\nu = \int_{\nu'+1/2=0}^{\nu'+1/2=-\infty} 2j_{-\nu'-1} \frac{(\nu'+1/2)\pi}{\sin \nu'\pi} e^{i(\nu'+1)\pi/2} P_{-\nu'-1} d\nu'.$$

But in general

$$(3.4) \qquad 2j_\nu e^{-i\nu\pi/2} = \left(h_\nu^{(1)} + h_\nu^{(2)}\right)e^{-i\nu\pi/2},$$

and, using the identities

$$(3.5) \qquad h_{-\nu'-1}^{(1)} = e^{i\pi(\nu'+1/2)} h_{\nu'}^{(1)}; \qquad h_{-\nu'-1}^{(2)} = e^{-i\pi(\nu'+1/2)} h_{\nu'}^{(2)}$$

and

$$(3.6) \qquad P_{-\nu'-1}(\cos \theta) = P_{\nu'}(\cos \theta) \qquad \text{for all } \nu',$$

we may write instead of (3.4)

$$(3.7) \qquad 2j_{-\nu'-1} e^{i(\nu'+1)\pi/2} = e^{-i\nu'\pi/2} 2j_{\nu'} - e^{+i\nu'\pi/2} h_{\nu'}^{(1)} 2 \cos \nu'\pi.$$

Now the integral in (3.3) becomes

$$\int_{\nu+1/2=0}^{\infty} \frac{(\nu+1/2)\pi}{\sin \nu\pi} e^{-i\nu\pi/2} 2j_\nu P_\nu d\nu = \int_{\nu'+1/2=0}^{-\infty} 2j_{\nu'} \frac{(\nu'+1/2)\pi}{\sin \nu'\pi} e^{-i\nu'\pi/2} P_{\nu'} d\nu'$$

$$(3.8)$$

$$- \int_{\nu'+1/2=0}^{-\infty} 2h_{\nu'}^{(1)} \frac{\cos \nu'\pi}{\sin \nu'\pi} e^{i\nu'\pi/2}(\nu'+1/2)\pi P_{\nu'} d\nu'.$$

The original integral about the positive real axis in the $\nu + \frac{1}{2}$ -plane now becomes an integral above the real axis from $+\infty$ to $-\infty$ plus an additional contour starting from the origin to $-\infty$. Removing the primes, we obtain for u_i

$$(3.9) \quad u_i = \frac{1}{2\pi i} \left[\int_{+\infty}^{-\infty} 2j_\nu \frac{(\nu+1/2)\pi}{\sin \nu\pi} e^{-i\nu\pi/2} P_\nu d\nu - \int_{\nu+1/2=0}^{-\infty} 2h_\nu^{(1)} \frac{\cos \nu\pi}{\sin \nu\pi} e^{i\nu\pi/2} (\nu+1/2)\pi P_\nu d\nu \right]$$
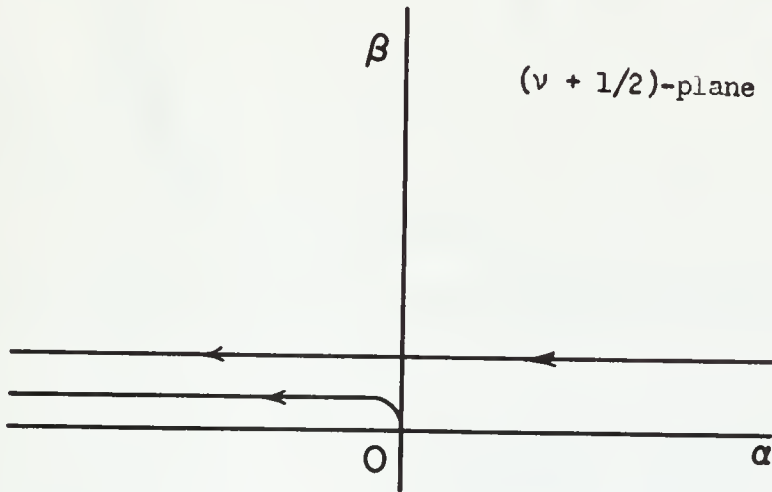
as illustrated in Fig. 3.

Figure 3

According to (2.7), there will be values of $\nu = \nu_n$ that specify the zeros of $B_{\nu_n}(ka)$. Assume that the ν_n's delineate a path C in the $\nu + \frac{1}{2}$ —plane as illustrated in Fig. 4; then deform the contours for u_i in the manner shown by the figure.
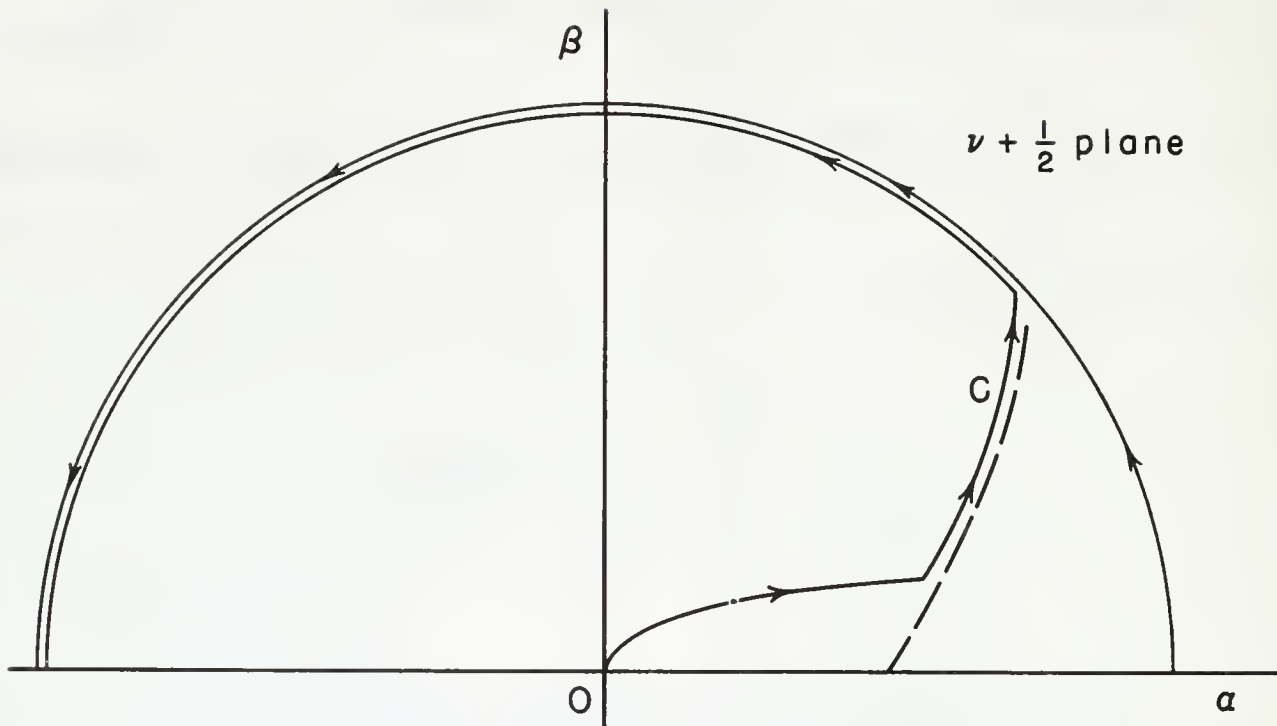
Figure 4

The integrals over the large semi-circle may be shown to vanish as the radius grows larger. Hence the integral representation for the incident wave may be written in terms of the integral whose contour starts at the origin and is deformed over the path delineated by the radial eigenvalues:

$$(3.10) \qquad u_i = +\frac{i}{2\pi} \int_C 2h_\nu^{(1)}(kr) \frac{\cos \nu\pi}{\sin \nu\pi} e^{i\nu\pi/2} (\nu+1/2)\pi P_\nu(\cos \Theta)d\nu \quad .$$

Now the scattered wave u_s is given by the total wave minus the incident wave. We write u_s as the sum for u given by (2.6) minus the integral in (3.10):

$$u_s = i\sum_{\nu_n} \left[\frac{A_\nu(ka)}{\frac{\partial}{\partial \nu} B_\nu(ka)} \right] \frac{(\nu_n + 1/2)\pi}{\sin \nu_n \pi} \exp\left[-\frac{i\pi}{2}(\nu_n + 1) \right] P_{\nu_n}(\cos \Theta) h_{\nu_n}^{(1)}(kr)$$

(3.11)

$$- \frac{i}{2\pi} \int_C 2h_\nu^{(1)}(kr) \frac{\cos \nu\pi}{\sin \nu\pi} e^{i\pi\nu/2} (\nu + 1/2)\pi \, P_\nu(\cos \Theta) \, d\nu \quad .$$

In the following sections we shall insert in (3.11) a change of variable in both the sum and the integral and introduce the tangent approximation for A_ν and B_ν in the sum. We recognize, though, that the tangent approximation to the Bessel functions is not valid for values of $\nu + 1/2$ near ka. Therefore, this approximation does not apply to the initial terms of the sum. After the first L terms however, when $\nu_L + 1/2$ is sufficiently far from ka, the tangent approximation becomes acceptable and may be used in the remaining terms of the sum. The criterion for determining L depends on the accuracy desired.

We intend to show that the tangent approximation for the sum effectively cancels the integral in (3.11) when its contour conforms to the path delineated by the eigenvalues. To achieve this, we specify $\nu_L + 1/2$ as the point at which the contour begins to conform with the delineated path (see Fig. 4).

In later sections, the initial terms of the sum are evaluated by the more accurate Hankel approximation. Also, the initial portion of the integral (from $\nu + 1/2$ equals zero to $\nu_L + 1/2$) is easily approximated if terms of order $(ka)^{-6/3}$ are neglected.

4. Evaluation of the total wave

Asymptotic forms for the spherical Hankel functions are obtained by the saddle-point method applied to the Sommerfeld integral representation for solutions to Bessel's equations. The forms that are appropriate here are

$$(4.1) \qquad h_\nu^{(1)}(ka) = \frac{2e^{-i\pi/4}}{ka} \left[\left(\frac{\nu + \frac{1}{2}}{ka} \right)^2 - 1 \right]^{-1/4} \cos\left(\frac{\pi}{4} + w\right) + \dots ,$$

$$(4.2) \qquad h_\nu^{(2)}(ka) = \frac{i}{ka} \left[\left(\frac{\nu + \frac{1}{2}}{ka} \right)^2 - 1 \right]^{-1/4} e^{-iw} + \dots ;$$

with the condition that the real and imaginary parts of $\nu + \frac{1}{2}$ are both greater than zero and that the function

$$(4.3) \qquad w = \sqrt{(ka)^2 - (\nu + \frac{1}{2})^2} - (\nu + \frac{1}{2}) \cos^{-1} \left(\frac{\nu + \frac{1}{2}}{ka} \right)$$

has a real part less than zero. Equations (4.1) and (4.2) are forms of the so-called tangent approximation. (cf. Bremmer [4]).

The derivatives of (4.1) and (4.2) are needed also. These are given by

$$(4.4) \quad h_\nu^{(1)'} = -2 \frac{e^{-i\pi/4}}{ka} \left[\left(\frac{\nu + 1/2}{ka} \right)^2 - 1 \right]^{-1/4} \sin\left(\frac{\pi}{4} + w\right) \frac{dw}{dka} + \text{a negligible term.}$$

$$(4.5) \quad h_\nu^{(2)'} = + \frac{1}{ka} \left[\left(\frac{\nu + 1/2}{ka} \right)^2 - 1 \right]^{-1/4} e^{-iw} \frac{dw}{dka} + \text{a negligible term .}$$

For brevity, let us introduce ξ and C defined by

$$(4.6) \qquad \frac{\partial w}{\partial ka} = C \sin \xi \quad \text{and} \quad Z = C \cos \xi .$$

Then B_ν reduces to

$$(4.7) \quad B_\nu = \left(h_\nu^{(1)'} - Z \, h_\nu^{(1)} \right) = -2 \frac{C \, e^{-i\pi/4}}{ka} \left[\left(\frac{\nu + 1/2}{ka} \right)^2 - 1 \right]^{-1/4} \cos(\xi - \frac{\pi}{4} - w) .$$

The zeros of B_ν occur when the cosine term in (4.7) vanishes. In a similar manner we find

$$(4.8) \quad A_\nu = 2j_\nu' - Z \, 2j_\nu = -\frac{iC}{ka} \left[\left(\frac{\nu + 1/2}{ka} \right)^2 - 1 \right]^{-1/4} e^{i(\xi - w)} + \left(h_\nu^{(1)'} - Z \, h_\nu^{(1)} \right) .$$

Now we introduce a change of variable. Let

$$(4.9) \qquad \xi - \frac{\pi}{4} - w = \pi(x + \frac{1}{2}) .$$

Since the real part of w must be less than zero, x is restricted to positive values. In particular, when x is zero or an integer, A_ν vanishes. This corresponds to $\nu = \nu_n$ provided the tangent approximation is valid. The derivative of B_ν with respect to ν is found from (4.7) to be

$$(4.10) \quad \frac{\partial}{\partial \nu} B_\nu = 2 \frac{C \, e^{-i\pi/4}}{ka} \left[\left(\frac{\nu + 1/2}{ka} \right)^2 - 1 \right]^{-1/4} \sin \pi(x + \frac{1}{2}) \frac{dx}{d\nu} \pi$$

$$+ \quad \text{a cosine term that vanishes at } \nu_n .$$

The ratio of (4.8) and (4.10) at $\nu = \nu_n$ becomes

$$(4.11) \quad \left[\frac{A_\nu}{\frac{\partial}{\partial \nu} B_\nu} \right]_{\nu_n} = \left[\frac{e^{i(x+1/2)\pi}}{2\pi \frac{dx}{d\nu} \sin(x + \frac{1}{2})\pi} \right]_{x=n} = \left[\frac{-1}{2\pi i \frac{dx}{d\nu}} \right]_{\nu=\nu_n}$$

The derivative $\frac{\partial w}{\partial \nu}$ is found from (4.3) to be

$$(4.12) \qquad \frac{\partial w}{\partial \nu} = -\cos^{-1}\left(\frac{\nu + \frac{1}{2}}{ka}\right) \quad .$$

Therefore, (4.11) reduces finally to

$$(4.13) \qquad \left[\frac{A_\nu}{\frac{\partial}{\partial \nu} B_\nu}\right]_{\nu_n} = \frac{i}{2 \cos^{-1}\left(\frac{\nu_n + 1/2}{ka}\right)} \quad .$$

Equation (2.6) for the total wave (or the sum in (3.11)) is written now as two sums. The first sum extends from ν_o to ν_L. The tangent approximation is not acceptable for the terms in this sum. The second sum, however, extends beyond ν_L; for this sum the tangent approximation, and therefore (4.13), is acceptable. Accordingly, we write u as

$$u = -\sum_{\nu_o}^{\nu_L} \left[\frac{A_\nu}{i \frac{\partial}{\partial \nu} B_\nu}\right]_{\nu_n} \frac{(\nu_n + 1/2) \exp\left[-\frac{i\pi}{2}(\nu_n + 1)\right]}{\sin \nu_n \pi} P_{\nu_n}(\cos \theta) h_{\nu_n}^{(1)}(kr)$$

$$(4.14)$$

$$-\sum_{\nu_{L+1}}^{\infty} \left[\frac{1}{2 \cos^{-1} \frac{\nu + 1/2}{ka}}\right]_{\nu_n} \frac{(\nu_n + 1/2) \exp\left[-\frac{i\pi}{2}(\nu_n + 1)\right]}{\sin \nu_n \pi} P_{\nu_n}(\cos \theta) h_{\nu_n}^{(1)}(kr)$$

5. Evaluation of the incident wave

The incident wave is given by (3.10). This is the integral subtracted from the series for the total wave in (3.11).

Let us write this integral in two parts, denoted by I_1 and I_2. The first integral, I_1, extends from the origin to the L-th eigenvalue, and the second integral, I_2, extends from the L-th eigenvalue outward. These paths are illustrated in Fig. 5.
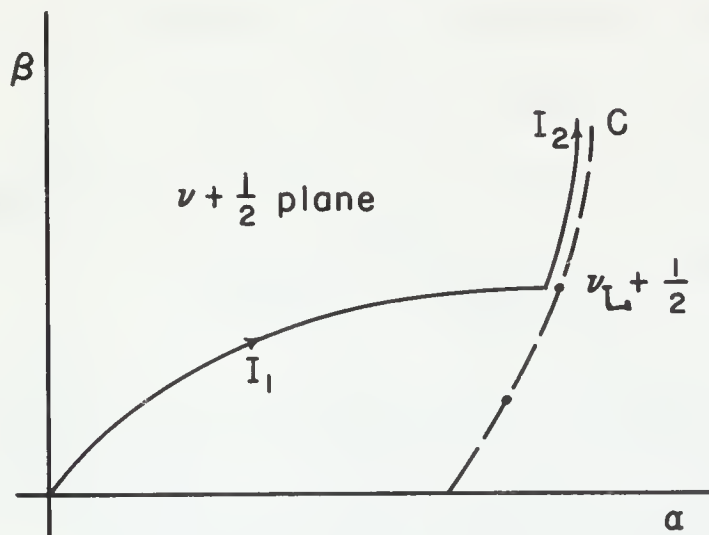

Figure 5

First consider the integral I_2. We introduce the change of variable defined in (4.9) and (4.3):

$$(5.1) \qquad w = \sqrt{(ka)^2 - (\nu + \tfrac{1}{2})^2} - (\nu + \tfrac{1}{2}) \cos^{-1}\left(\frac{\nu + \tfrac{1}{2}}{ka}\right) \equiv -\pi(x + \tfrac{3}{4}) + \xi .$$

Then

$$(5.2) \qquad dw = \frac{\partial w}{\partial \nu} d\nu = -\pi \, dx .$$

$\frac{\partial w}{\partial \nu}$ has already been found from (4.12). Substituting this value into (5.2) we have

$$(5.3) \qquad d\nu = \frac{\pi \, dx}{\cos^{-1}\left(\dfrac{\nu + 1/2}{ka}\right)} \quad .$$

Next, we use an asymptotic form for $\cos \nu\pi$:

$$(5.4) \qquad \cos \nu\pi = \frac{1}{2} e^{-i\nu\pi}\left(1 + e^{+2i\nu\pi}\right) = \frac{1}{2} e^{-i\nu\pi} + \dots \quad .$$

In a later section, we will demonstrate that the imaginary part of ν_L is like

$$\operatorname{Im} \nu_L \sim (-3 \, w_L)^{2/3} \, (ka)^{1/3} \; .$$

Then, for any value for $\operatorname{Im} \nu$ along the path taken by I_2, the asymptotic term in (5.4) is smaller than

$$\varepsilon = 0 \, e^{-(ka)^{1/3} \, (-3 \, w_L)^{2/3}} \; ,$$

which approaches zero faster than any negative power of ka as ka approaches infinity. When ka is as low as unity, neglecting the asymptotic term results in an error of less than 1 part in 20,000 at the L-th eigenvalue.

Using (5.3) and the asymptotic form of (5.4) in the integral (3.10), we may write I_2 as

$$(5.5) \qquad I_2 = -\frac{1}{2} \int_{x=L}^{\infty} \frac{(\nu + 1/2)\pi}{\cos^{-1}\left(\dfrac{\nu + 1/2}{ka}\right)} \cdot \frac{e^{-i\pi(\nu+1)/2}}{\sin \nu\pi} \, P_\nu(\cos \theta) h_\nu^{(1)}(kr) \, dx \quad .$$

If we consider the change of variables introduced earlier, i.e., compare (5.1) with (4.3) et seq., we see that x is a real variable along the path prescribed for I_2 and that ν takes on the values ν_n whenever x is an integer.

Let us now evaluate the integral (5.5) by the Euler-Maclaurin sum rule (see Whittaker and Robinson [5]). This series is the following expression for a definite integral:

$$(5.6) \quad \int_{x=a}^{x=a+N\triangle x} f(x)dx = \left[\frac{1}{2} \Big(f(a) - f(a+N\triangle x) \Big) + \sum_{n=1}^{n=N} f(a+n\triangle x) \right] \triangle x + \delta \ .$$

This expression is equivalent to truncating the curve for $f(x)$ as shown in Fig. 6.

$f(x)$

$\triangle x$

a

x

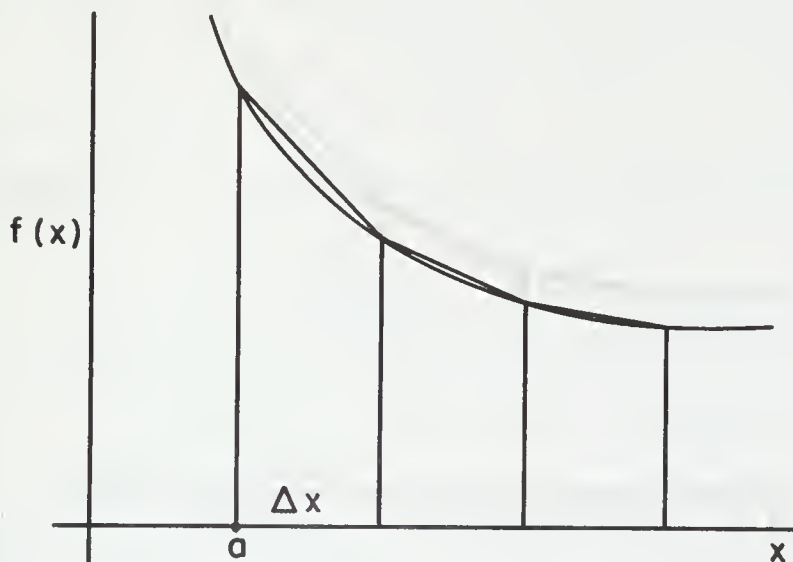Figure 6

across successive values of $\triangle x$ and summing the areas of the trapezoids. The term δ in (5.6) is a correction term for the error resulting from the truncations and is given by

$$(5.7) \quad \delta = \sum_{m=0}^{\infty} B_{m+1} \frac{(-1)^{m+1}\triangle x^{2m+1}}{(2m+2)!} \left[\frac{d^{2m+1}}{dx^{2m+1}} f(a+N\triangle x) - \frac{d^{2m+1}}{dx^{2m+1}} f(a) \right] \ .$$

The coefficients B_{2m+1} are called the Bernoullian numbers. The first few of these are

$$(5.8) \qquad B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad \cdots \quad .$$

For our purposes, $a = L$ and $\triangle x = 1$, which corresponds to the interval between the successive eigenvalues ν_n. Also it may be shown that the integrand in (5.5) as well as its derivatives vanish as N approaches infinity. For the case at hand, then, the Euler-Maclaurin rule becomes

$$(5.9) \qquad \int_{x=L}^{\infty} f(x)dx = \frac{1}{2} f(L) + \sum_{n=L+1}^{\infty} {}' f(n) + \delta ,$$

where f is the integrand of (5.5). Accordingly, δ becomes

$$(5.10) \qquad \delta = \sum_{m=0}^{\infty} B_{m+1} \frac{(-1)^m}{(2m+2)!} \frac{d^{2m+1}}{dx^{2m+1}} f(L) .$$

I_2 may now be written as

$$(5.11) \quad I_2 = \frac{1}{2} f(L) + \left[-\frac{1}{2} \sum_{n=L+1}^{\infty} \frac{(\nu_n + 1/2)\pi}{\cos^{-1}\left(\frac{\nu_n + 1/2}{ka}\right)} \frac{e^{-i\pi(\nu_n+1)/2}}{\sin \nu_n \pi} P_{\nu_n}(\cos \theta) h_{\nu_n}^{(1)}(kr) \right] + \delta .$$

Comparison with equation (4.14) for the total wave shows that the sum appearing in (5.11) agrees exactly with that in (4.14) for $n=L+1$ and beyond. Thus I_2 may be written as

$$(5.12) \quad I_2 = \sum_{\nu_0}^{\nu_L} \left[\frac{A_\nu}{i \frac{\partial}{\partial \nu} B_\nu} \right]_{\nu_n} \frac{(\nu_n + 1/2)\pi \cdot \exp\left[-\frac{i\pi}{2}(\nu_n + 1) \right]}{\sin \nu_n \pi} P_{\nu_n}(\cos \theta) h_{\nu_n}^{(1)}(kr)$$

$$+ u + \frac{1}{2} f(L) + \delta ,$$

and the incident wave may be expressed by

$$u_i = u + \sum_{\nu_o}^{\nu_L} \left[\frac{A_\nu}{i \frac{\partial}{\partial \nu} B_\nu} \right]_{\nu_n} \frac{(\nu_n + 1/2)\pi \exp\left[-\frac{i\pi}{2}(\nu_n + 1)\right]}{\sin \nu_n \pi} P_{\nu_n}(\cos\theta) h_{\nu_n}^{(1)}(kr)$$

(5.13)
$$+ I_1 + \frac{1}{2} f(L) + \delta .$$

(5.13) shows that the difference between u and u_i i.e., the scattered wave, consists of a finite series, a definite integral, and a correction.

6. Evaluation of the scattered wave in the forward direction

In order to evaluate the definite integral I_1, we introduce certain restrictions. First we consider only the far field, for which r >> a. Then $h_{\nu_n}^{(1)}(kr)$ may be written asymptotically as (cf. Stratton[3])

(6.1)
$$h_\nu^{(1)} = \frac{e^{ikr}}{kr} \cdot e^{-i\pi(\nu+1)/2} .$$

We also restrict the problem to forward scattering, for which

(6.2)
$$P_\nu(1) = 1 \quad \text{for all } \nu .$$

Then the integral I_1, which is (3.10) extended from zero to $\nu_L + 1/2$, can be expressed as

(6.3)
$$I_1 = -\frac{1}{\pi^2} \frac{e^{ikr}}{kr} \int_0^{(\nu_1 + 1/2)} \tan(\nu + \frac{1}{2})\pi \cdot (\nu + \frac{1}{2})\pi \, d\nu\pi$$

This integral is most easily evaluated along the path shown in Fig. 7.
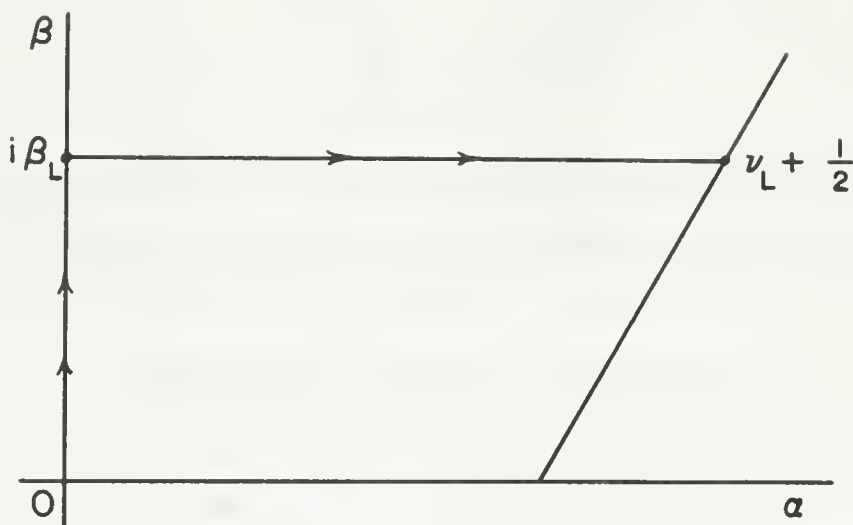
Figure 7

Let us denote the $\text{Im}(\nu + \frac{1}{2})$ by β; then the integral along the imaginary axis is

(6.4) $$\int_0^{\beta_1 \pi} \tan(\nu + \tfrac{1}{2})\pi (\nu + \tfrac{1}{2})\pi \, d\nu\pi = -i \int_0^{\beta_1} \tanh \beta\pi \, \beta\pi \, d\beta\pi .$$

A graph of $\tanh \beta\pi \cdot \beta\pi$ appears in Fig. 8; the shaded area shows how it departs from a straight line. This shaded area is finite no matter
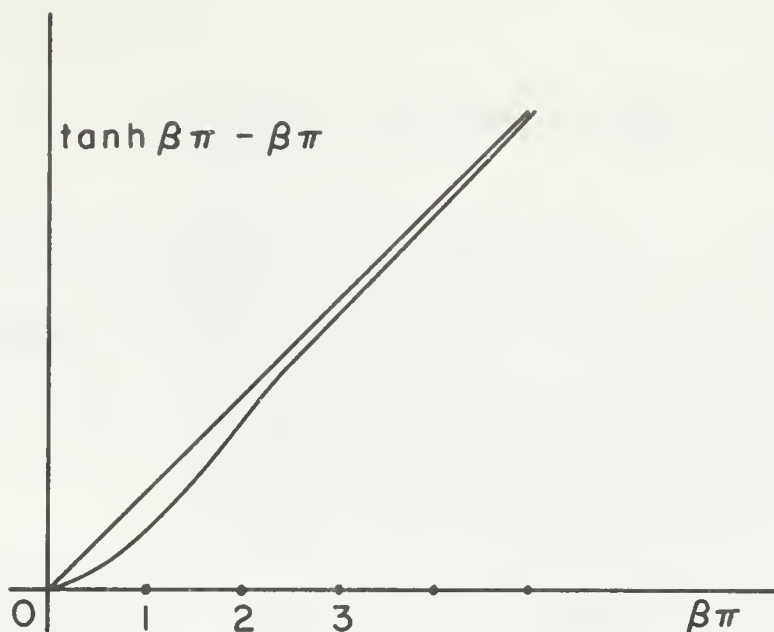
Figure 8

how far $\beta\pi$ is extended. Let us call this area $\triangle A$, and define it in terms of (6.4) as

$$(6.5) \qquad - i \int_0^{\beta_1\pi} \tanh \beta\pi \; \beta\pi \; d\beta\pi \; = \; - i \int_0^{\beta_1\pi} \beta\pi \; d\beta\pi + i \triangle A(\beta_1\pi) \; .$$

For that portion of the integral extending from $i\beta_L$ to $\nu_L + \frac{1}{2}$ shown in Fig. 7, the asymptotic forms for the cosine (and sine) specified by (5.4) may be used, yielding

$$\tanh(\nu + \tfrac{1}{2})\pi \; = \; i \left(1 + o\left(e^{-(ka)^{1/3}} \right) \right) \; .$$

Therefore, the entire integral I_1 may be written

$$(6.6) \qquad I_1 \; = \; - \frac{i}{\pi^2} \; \frac{e^{ikr}}{kr} \left[\int_0^{(\nu_L + 1/2)} (\nu + \tfrac{1}{2})\pi \; d\nu\pi + \triangle A \right] \; ,$$

which reduces immediately to

$$(6.7) \qquad I_1 = - i \frac{e^{ikr}}{kr} \left[\frac{(\nu_L + 1/2)^2}{2} + \frac{\Delta A}{\pi^2} \right] .$$

We turn now to evaluating the finite series given in (5.13). To this end, we write first the Hankel approximation for $h_\nu^{(1)}$, $2j_\nu$, and their derivatives (cf. Bremmer[4])

$$(6.8) \qquad h_\nu^{(1)}(ka) = \frac{1}{ka} \sqrt{\frac{2\pi w}{3p}} \left[J_{1/3}(-w) + J_{-1/3}(-w) \right]$$

$$(6.9) \qquad 2j_\nu(ka) = \frac{1}{ka} \sqrt{\frac{2\pi w}{3p}} \left[e^{-i\pi/3} J_{1/3}(-w) + e^{i\pi/3} J_{-1/3}(-w) \right]$$

$$(6.10) \qquad h_\nu^{(1)\prime}(ka) = \frac{1}{ka} \sqrt{\frac{2\pi w}{3p}} \left[J_{2/3}(-w) - J_{-2/3}(-w) \right] \cdot \left(\frac{dw}{dka} \right)$$

$$(6.11) \qquad 2j_\nu^\prime(ka) = \frac{1}{ka} \sqrt{\frac{2\pi w}{3p}} \left[e^{i\pi/3} J_{2/3}(-w) - e^{-i\pi/3} J_{-2/3}(-w) \right] \cdot \left(\frac{dw}{dka} \right) .$$

In these equations, w has been given before by (4.3), and p is given by

$$p = \sqrt{1 - \frac{\nu + 1/2}{ka}^2} .$$

Note also that (6.8) and (6.10) can have zeros only for negative real values of w.

Let us consider the Dirichlet condition first, letting $z \rightarrow \infty$. This reduces the bracketed term of the sum in (5.13) to

$$(6.12) \qquad \left[\frac{A_\nu}{i \frac{\partial}{\partial \nu} B_\nu} \right]_{\nu_n} = \left[\frac{2j_\nu(ka)}{i \frac{\partial}{\partial \nu} h_\nu^{(1)}(ka)} \right]_{\nu_n}$$

where the ν_n are obtained from $h_{\nu_n}^{(1)}(ka) = 0$. (6.8) shows that these roots follows from

$$(6.3) \qquad\qquad J_{1/3}(-w_n) = - J_{-1/3}(-w_n) \ .$$

We need as well

$$\frac{\partial}{\partial \nu} h_\nu^{(1)}(ka) = - \frac{1}{ka} \sqrt{\frac{2\pi w}{3p}} \left[J'_{1/3}(-w_n) + J'_{-1/3}(-w_n) \right] \left(\frac{\partial w}{\partial \nu} \right)_{\nu_n}$$

(6.14)

$$+ \text{ a vanishing term at } \nu_n \ .$$

(6.13), (6.14) and (6.9) reduce (6.12) to

$$(6.15) \quad \left[\frac{A_\nu}{i \frac{\partial}{\partial \nu} B_\nu} \right]_{\nu_n} = \frac{\sqrt{3} \ J_{1/3}(-w_n)}{\left(J'_{1/3}(-w_n) + J'_{-1/3}(-w_n) \right) \left(\frac{\partial w}{\partial \nu} \right)_{\nu_n}} = \frac{- D_n}{\left(\frac{\partial w}{\partial \nu} \right)_{\nu_n}}$$

where

$$(6.16) \quad D_n = \frac{- \sqrt{3} \ J_{1/3}(-w_n)}{J'_{1/3}(-w_n) + J'_{-1/3}(-w_n)} \ , \qquad \text{subject to (6.13) } \ .$$

We call D_n the Dirichlet coefficient, where, according to (4.11), D_n approaches $\frac{1}{2}$ as the tangent approximation becomes acceptable.

The Neumann boundary condition for which $z = 0$ is treated in an analogous manner. For this condition

$$(6.17) \quad \left[\frac{A_\nu}{i \frac{\partial}{\partial \nu} B_\nu} \right]_{\nu_n} = \left[\frac{2 j'_\nu (ka)}{i \frac{\partial}{\partial \nu} h_\nu^{(1)'}(ka)} \right]_{\nu_n}$$

where the ν_n are obtained from $h_{\nu_n}^{(1)'}(ka) = 0$. (6.10) shows that these are the roots of

$$(6.18) \qquad\qquad J_{2/3}(-w_n) = J_{-2/3}(-w_n) \ .$$

The derivative of (6.10) required for (6.17) is

$$\frac{\partial}{\partial \nu} h_\nu^{(1)'}(ka) = -\frac{1}{ka} \sqrt{\frac{2\pi w}{3p}} \left[J_{2/3}'(-w_n) - J_{-2/3}'(-w_n) \right] \left(\frac{\partial w}{\partial \nu}\right)_{\nu_n}$$

(6.19)

$$+ \text{ a vanishing term at } \nu_n .$$

(6.11), (6.18) and (6.19) reduce (6.17) to

(6.20)
$$\left[\frac{A_\nu}{i \frac{\partial}{\partial \nu} B_\nu} \right]_{\nu_n} = \frac{-\sqrt{3} \, J_{-2/3}(-w_n)}{\left[J_{2/3}'(-w_n) - J_{-2/3}'(-w_n) \right] \cdot \left(\frac{\partial w}{\partial \nu}\right)_{\nu_n}} = \frac{-N_n}{\left(\frac{\partial w}{\partial \nu}\right)_{\nu_n}}$$

where

(6.21)
$$N_n = \frac{\sqrt{3} \, J_{-2/3}(-w_n)}{J_{2/3}'(-w_n) - J_{-2/3}'(-w_n)} , \quad \text{subject to } J_{2/3}(-w_n) = J_{-2/3}(-w_n) .$$

We call the N_n the Neumann coefficient, which must also approach $\frac{1}{2}$ as the tangent approximation becomes acceptable. Through the use of (6.1), (6.2), the asymptotic form for $\sin \nu \pi$, and either (6.15) or (6.20), the sum appearing in (5.13) reduces to

(6.22)
$$- 2\pi i \frac{e^{ikr}}{kr} \sum C_n \frac{(\nu_n + 1/2)}{\left(\frac{\partial w}{\partial \nu}\right)_{\nu_n}} ,$$

where C_n is either D_n or N_n as the case may be.

Also, the correction term δ must be evaluated. This is obtained from (5.10), where the function f is the integrand of (5.5). After inserting the appropriate asymptotic forms and specializing to the forward direction, f becomes

(6.23)
$$f = i\pi \frac{e^{ikr}}{kr} \frac{(\nu + 1/2)}{\left(\frac{\partial w}{\partial \nu}\right)} .$$

In (6.23), equation (4.12) has also been used.

In both (6.22) and (6.23), the term

$$(6.24) \qquad \frac{\nu + 1/2}{\partial w/\partial \nu}$$

appears. We now obtain an approximation for this term. This is done by inverting (4.3), which yields (cf. Bremmer[4])

$$(6.25) \qquad \nu + 1/2 = ka \left[1 + \frac{1}{2} (ka)^{-2/3}(-3w)^{2/3} e^{i\pi/3} + O(ka)^{-4/3} \right].$$

$\partial w/\partial \nu$ is found by differentiating (6.25):

$$(6.26) \qquad \frac{\partial w}{\partial \nu} = - \left[(-3w)^{-1/3}(ka)^{1/3} e^{i\pi/3} + O(ka)^{-4/3} \right]^{-1}.$$

Therefore (6.24) becomes

$$(6.27) \qquad \frac{\nu + 1/2}{\partial w/\partial \nu} = -(ka)^2 \left[(ka)^{-2/3}(-3w)^{-1/3} e^{i\pi/3} + O(ka)^{-4/3} \right].$$

The integral I_1, given by (6.7), may be expressed at once through (6.25) as

$$(6.28) \quad I_1 = - i \frac{e^{ikr}}{kr} \frac{(ka)^2}{2} \left[1 + (ka)^{-2/3}(-3w_L)^{2/3}e^{i\pi/3} + O(ka)^{-4/3} + \frac{2\Delta A}{\pi^2} (ka)^{-6/3} \right].$$

Substituting (6.27) into the series given in (6.22) yields

$$(6.29) \qquad 2\pi i \frac{e^{ikr}}{kr} (ka)^2 e^{i\pi/3} \left[\sum_{\nu_o}^{\nu_L} C_n (ka)^{-2/3}(-3w_n)^{-1/3} + O(ka)^{-4/3} \right].$$

Also (6.23), again by (6.27) becomes

$$(6.30) \quad f = - i\pi \frac{e^{ikr}}{kr} (ka)^2 \left[(ka)^{-2/3}(-3w)^{-1/3} e^{i\pi/3} + O(ka)^{-4/3} \right].$$

The correction term (see (5.10)) requires the derivative of f with

respect to x at $\nu_L + 1/2$. This is done through (4.9):

$$(6.31) \qquad \frac{d^{2m+1}}{dx^{2m+1}} (-3w)^{-1/3} = \frac{d^{2m+1}}{dx^{2m+1}} \left[3\pi(x + 3/4) - 3\xi \right]^{-1/3} ,$$

where $\xi = 0$ for the Dirichlet condition, and $\pi/2$ for the Neumann condition. The asymptotic form for the scattered wave in the forward direction may be written now, using (5.13) and the preceding equations:

$$u_s = u - u_i = i \frac{e^{ikr}}{kr} \frac{(ka)^2}{2} \left\{ 1 + (ka)^{-2/3} e^{i\pi/3} \left[(-3w_L)^{2/3} + (-3w_L)^{-1/3} \cdot \pi \right. \right.$$

$$(6.32) \qquad \left. - 4\pi \sum_{\nu_o}^{\nu_L} C_n (-3w_n)^{-1/3} + 2\pi \sum_{m=0}^{\infty} B_{m+1} \frac{(-1)^m}{(2m+2)!} \right.$$

$$\left. \left. \cdot \frac{d^{2m+1}}{dx^{2m+1}} \left[3\pi(x + 3/4) - 3\xi \right]^{-1/3} \right] \right\}$$

where higher-order term have been omitted.

We shall compute now the values of (6.32) for the two boundary conditions. Consider the Dirichlet condition first. We compare the roots found from the Hankel approximation (6.13) with those of the tangent approximation, namely the zeros of $\cos(\pi/4 + w)$ in (4.1):

	Tangent approx.	Hankel approx.
$-w_0$	2.35619	2.38346
$-w_1$	5.49778	5.51020
$-w_2$	8.63937	8.64736
$-w_3$	11.78096	11.78685

The last values quoted above differ by about 1 part in 2000. The values of D_n (6.16) corresponding to the roots from the Hankel approximation are given in the next table:

	D_n
w_0	.49495
w_1	.49895
w_2	.49959
w_3	.49980

These values clearly approach 1/2, the last term differing by about 2 parts in 5000. The correction term is found from the first term of the series for δ to be

$$\delta = -.00709 .$$

Higher corrections give contributions beyond the fifth decimal place.

After the remaining arithmetical computation is performed, we obtain

$$(6.33) \qquad u_s(\theta = 0) = i \frac{e^{ikr}}{kr} (ka)^2 \left[\frac{1}{2} + (ka)^{-2/3} e^{i\pi/3} \times .99668 \right]$$

for the Dirichlet condition.

We turn finally to the Neumann condition. First, we compare the roots determined by the Hankel approximation (6.18) with those of the tangent approximation, i.e., the zeros of $\sin(\pi/4 + w)$ in (4.4):

	Tangent approx.	Hankel approx.
$-w_0$	0.78540	0.68542
$-w_1$	3.92699	3.90279
$-w_2$	7.06858	7.05494
$-w_3$	10.21018	10.20068

The last terms differ by about 1 part in 1000.

The values of N_n (6.21) obtained from the Hankel approximation are found to be

	N_n
w_0	.54978
w_1	.50295
w_2	.50093
w_3	.50044

which also approaches $1/2$, but in the opposite sense from the D_n's. The correction term is found again from the first term of the series for δ to be

$$\delta = - .00861$$

and the remaining computation results in

$$(6.34) \qquad u_s(\theta = 0) = i \frac{e^{ikr}}{kr} (ka)^2 \left[\frac{1}{2} - (ka)^{-2/3} e^{i\pi/3} .8654 \right]$$

for the Neumann condition.

The total scattering cross section may be found by multiplying the imaginary part of the scattered amplitude by $4\pi/k$ (cf. Bohm[6]). The final results are

$$(6.35) \qquad \sigma = 2\pi a^2 \left(1 + (ka)^{-2/3} .99668\right); \text{ Dirichlet case },$$

$$(6.36) \qquad \sigma = 2\pi a^2 \left(1 - (ka)^{-2/3} .8654\right); \text{ Neumann case }.$$

These results are in substantial agreement with those found recently by Wu and Rubinow[7], who obtain values of .99618 and -.8640 for the two values above.

It is of interest to note that if the tangent approximation is used throughout (which simplifies the procedure outlined in this report) an error of approximately 3 percent is found for the Dirichlet condition, but considerably greater error, about 30 percent, occurs for the Neumann condition.

7. Conclusion

The significant feature of the method employed in this problem is the manner of subtracting the plane wave. It was shown that the scattered wave could be represented by a finite series plus a definite integral (apart from the correction δ). (5.13) shows that this can be done without restricting the general impedance boundary conditions or the values of r and θ. Also, it can be shown that the subtraction applies for the source located at a finite position; one need not restrict the method to an incoming plane wave.

It is necessary to depart from generality only to evaluate the definite integral. This would not be necessary for the sum, since appropriate approximations exist for P_ν as well as for $h_\nu^{(1)}$. Therefore, extension to the more general case appears plausible and deserves further study.

The distinction between the cases treated here and more general cases lies in locating the singular points, hence the radial eigenvalues. But the form and accuracy of the solution would be comparable.

Acknowledgments

References

[1] Friedman, B. — The dipole field in an inhomogeneous atmosphere; Comm. Pure and Appl. Math., $\underline{4}$, 317 (1951).

Franz, W. — Ueber die Greenschen Funktionen des Zylinders und der Kugel; Zeit f. Naturfor., $\underline{9a}$, 705-716 (1954).

[2] Marcuvitz, N. — Field representations in spherically stratified regions; Comm. Pure and Appl. Math., $\underline{4}$, 263 (1951).

[3] Stratton, J.A. — Electromagnetic Theory; McGraw-Hill, 1941. See Ch. VII, esp. p. 406.

[4] Bremmer, H. — Terrestial Radio Waves; Elsevier Publishing Co., New York-Houston, 1949. See Ch. II-IV, esp. p. 45.

[5] Whittaker, E. and Robinson, G. — The Calculus of Observations, 4th Edition, Blackie and Son, Ltd., 1944. Ch. VII, p. 134.

[6] Bohm, D. — Quantum Theory, Prentice-Hall Inc., 1951. Ch. XXI, pp. 564-565. Compare the imaginary part of equation (79) with equation (8.1a).

[7] Wu, T. and Rubinow, S. — The correction to geometric-optical cross sections of circular cylinders and spheres; Scientific Report No. 3, Cruft Laboratory, Harvard University, 1955.

White, F.P. — Diffraction of Plane Electromagnetic Waves by a Perfectly Reflecting Sphere; Proc. Roy. Soc. (A), $\underline{100}$, 505 - 525, (1922).

Date Due

OCT 23 57			
MAY 20 '65			
	PRINTED	IN U. S. A.	